An Introduction to Control Barrier Function

Theory and Application

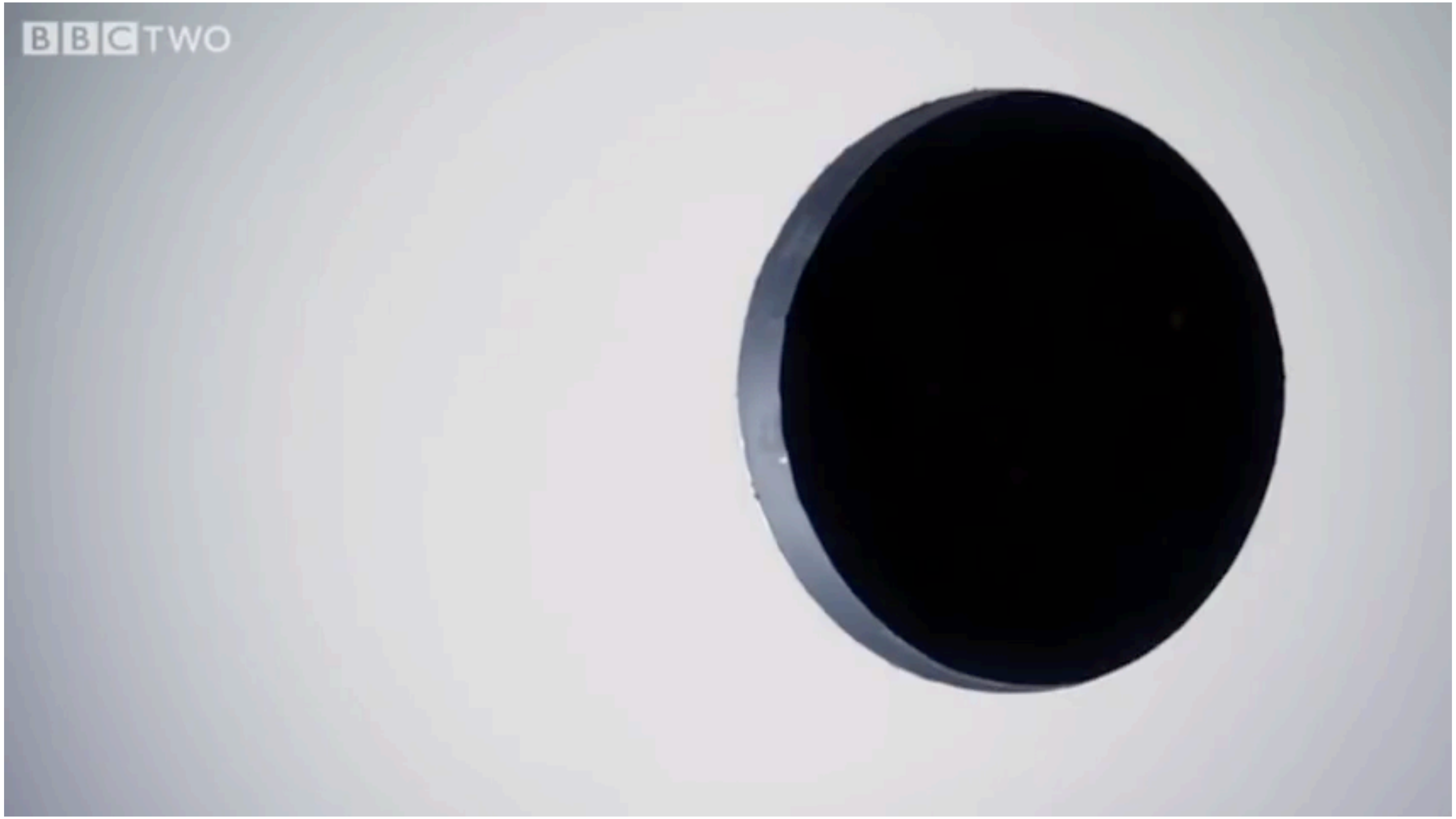
Bolun Dai

Motivation

Safe Control

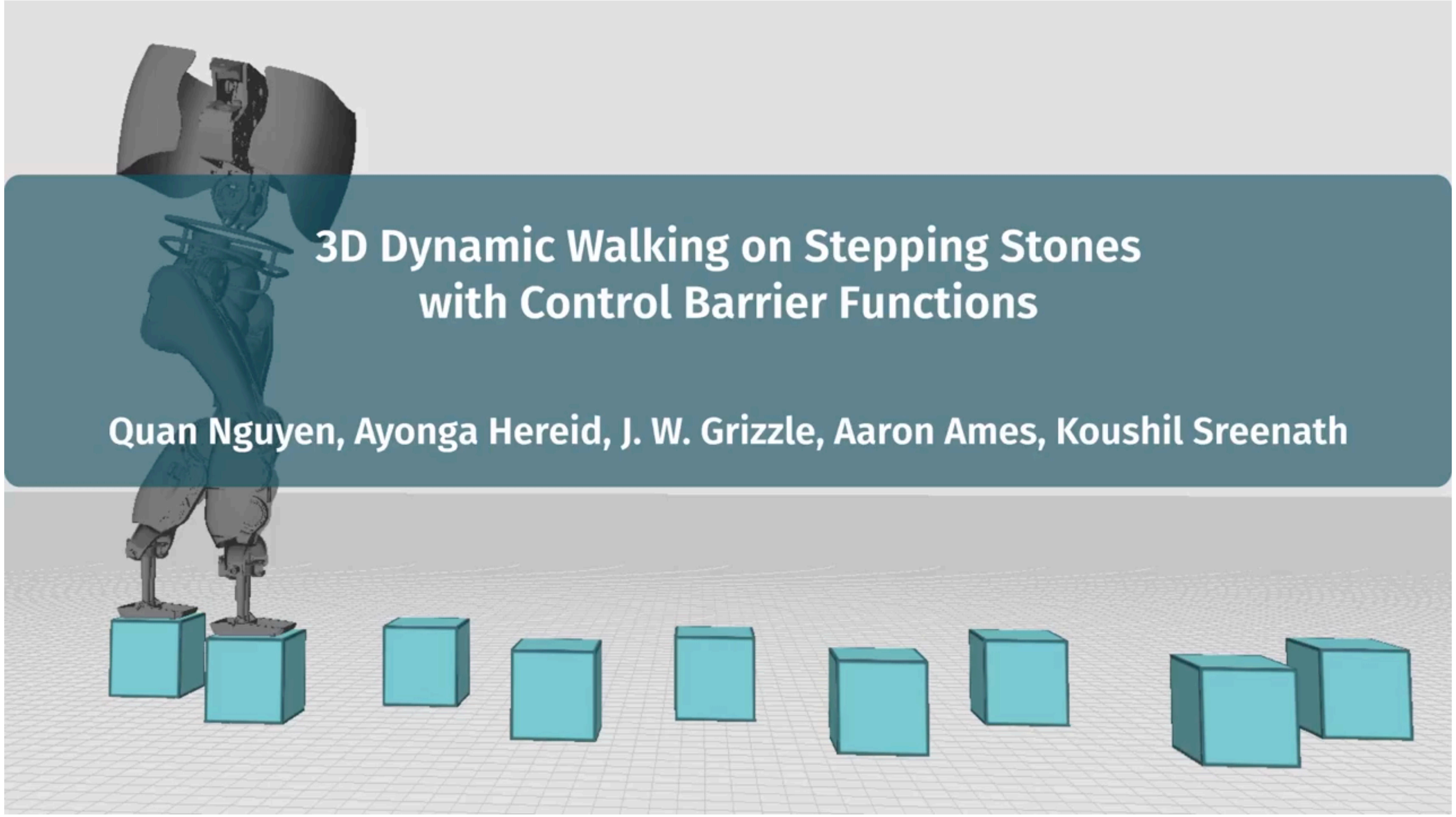
Motivation

Agile Behavior Under Constraints



Applications

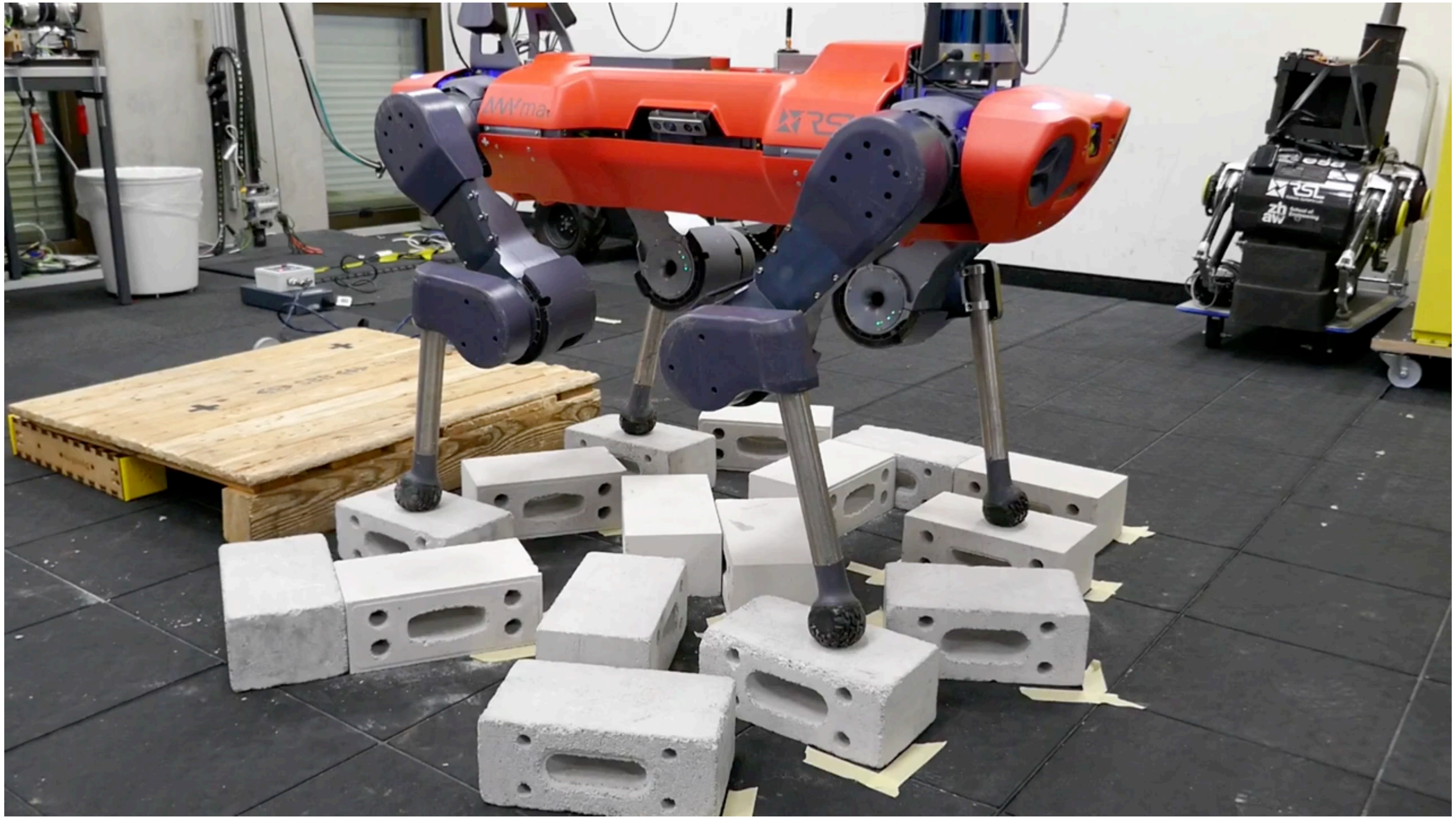
Bipedal Locomotion



Nguyen et al (2016). 3D dynamic walking on stepping stones with control barrier functions.

Applications

Quadruped Locomotion



Grandia et al (2021). Multi-Layered Safety for Legged Robots via Control Barrier Functions and Model Predictive Control

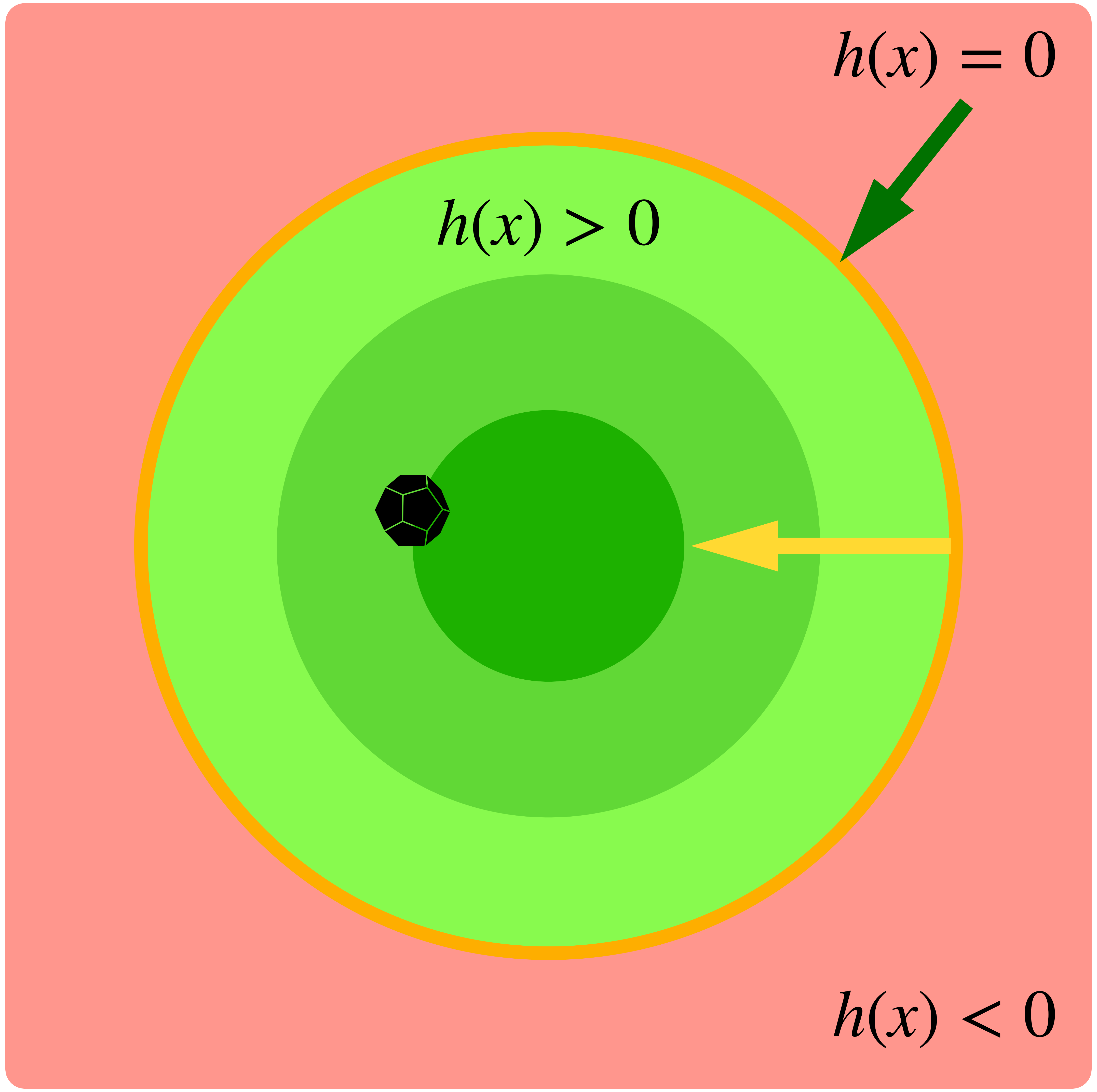
Structure of this talk

- Control Barrier Function (CBF)
- CLF-CBF-QP
- CBF Example
- Exponential Control Barrier Function (ECBF)
- ECBF Example
- CBF Research

Control Barrier Function (CBF)

Nagumo's Invariance Principle

- $\mathscr{C} = \{x \in \mathbb{R}^n \mid h \geq 0\}$
- $\dot{h}(x) \geq 0, \forall x \in \partial\mathscr{C}$



Control Barrier Function (CBF)

Nagumo's Invariance Principle

- Given a dynamical system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$, and assume that the safe set $\mathscr{C}$ is the superlevel set of a smooth function $h : \mathbb{R}^n \to \mathbb{R}$,

$$\mathscr{C} = \{x \in \mathbb{R}^n \mid h \geq 0\}$$

- then $\mathscr{C}$ is forward invariant if and only if $\dot{h}(x) \geq 0$ for all $x \in \partial\mathscr{C}$.

Nagumo, M. (1942). Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen.

Control Barrier Function (CBF)

Control Affine Systems

- Control affine systems have the form of

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$$

- where $\mathbf{x} \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $\mathbf{u} \in \mathbb{R}^m$. Control affine system are very common, most mechanical systems are control affine

$$\begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{q}} \\ M^{-1}(C\dot{\mathbf{q}} + G) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ M^{-1} \end{bmatrix} \mathbf{u}$$

Control Barrier Function (CBF)

Control Barrier Function

- For control affine systems $\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$, we have

$$\dot{h}(x) = \frac{\partial h(x)}{\partial x}\dot{x} = \frac{\partial h(x)}{\partial x}\Big(f(x) + g(x)u\Big) = L_f h(x) + L_g h(x)u$$

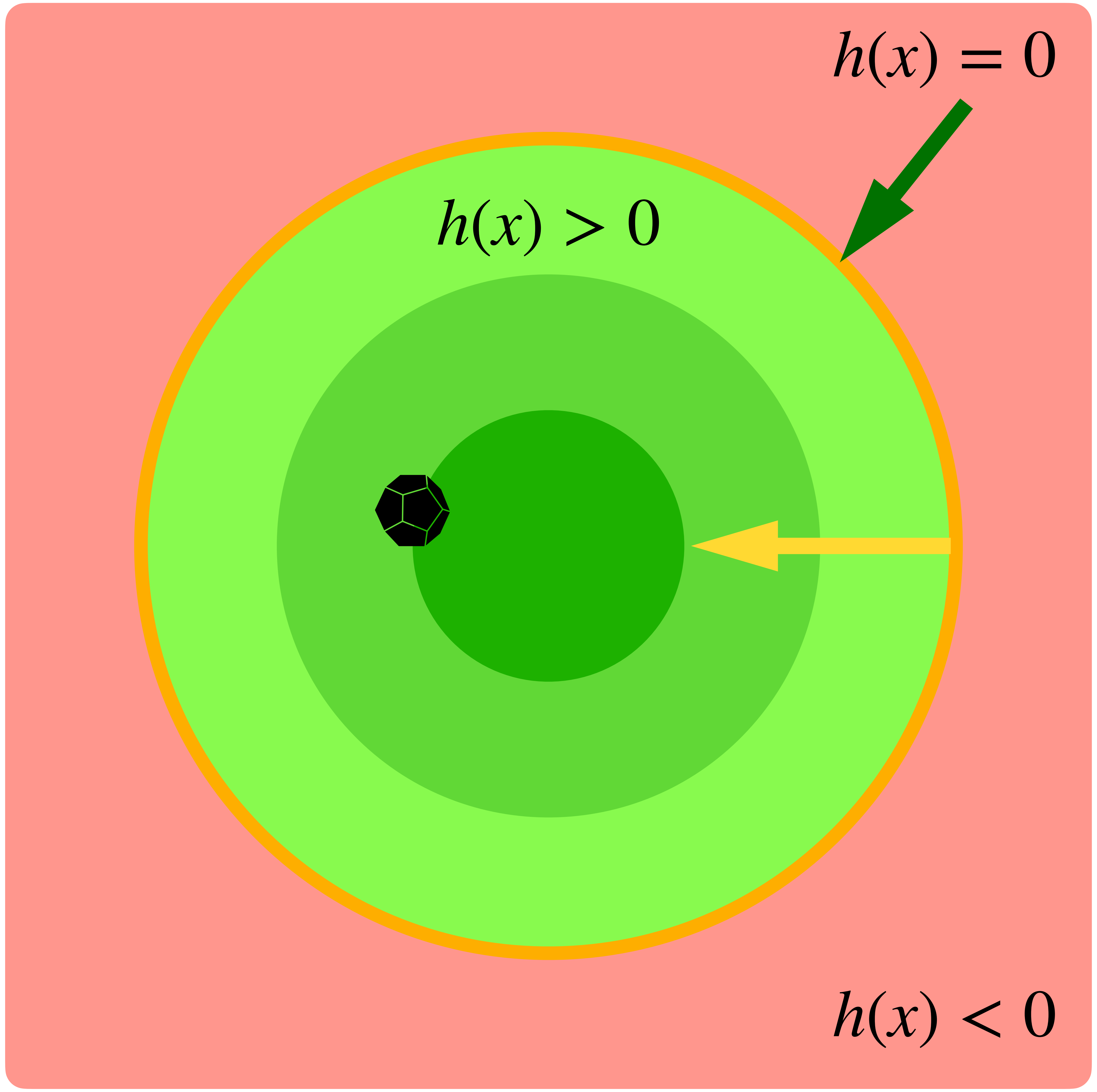
- which can be written using Lie derivatives

$$L_f h(x) = \frac{\partial h(x)}{\partial x} f(x),\ L_g h(x) = \frac{\partial h(x)}{\partial x} g(x)$$

Control Barrier Function (CBF)

Finding a control constraint using $h(x)$

- What are the issues of using $\dot{h}(x) \geq 0, \forall x \in \partial\mathscr{C}$ as a control constraint?
  - Abrupt behavior at the boundary, large control action.
- What are the issues of using $\dot{h}(x) \geq 0, \forall x \in \mathscr{C}$?
  - Too restrictive.



Control Barrier Function (CBF)

CBF Constraint

- For safe control, we can define a safe set $\mathscr{C}$, such that for a function $h(\mathbf{x})$ it is always positive

$$\mathscr{C} = \{\mathbf{x} \mid h(\mathbf{x}) \geq 0\}$$

- If we can find a control $\mathbf{u}$, such that the safe set $\mathscr{C}$ is forward invariant, we then have a valid CBF. This condition can be expressed using the inequality

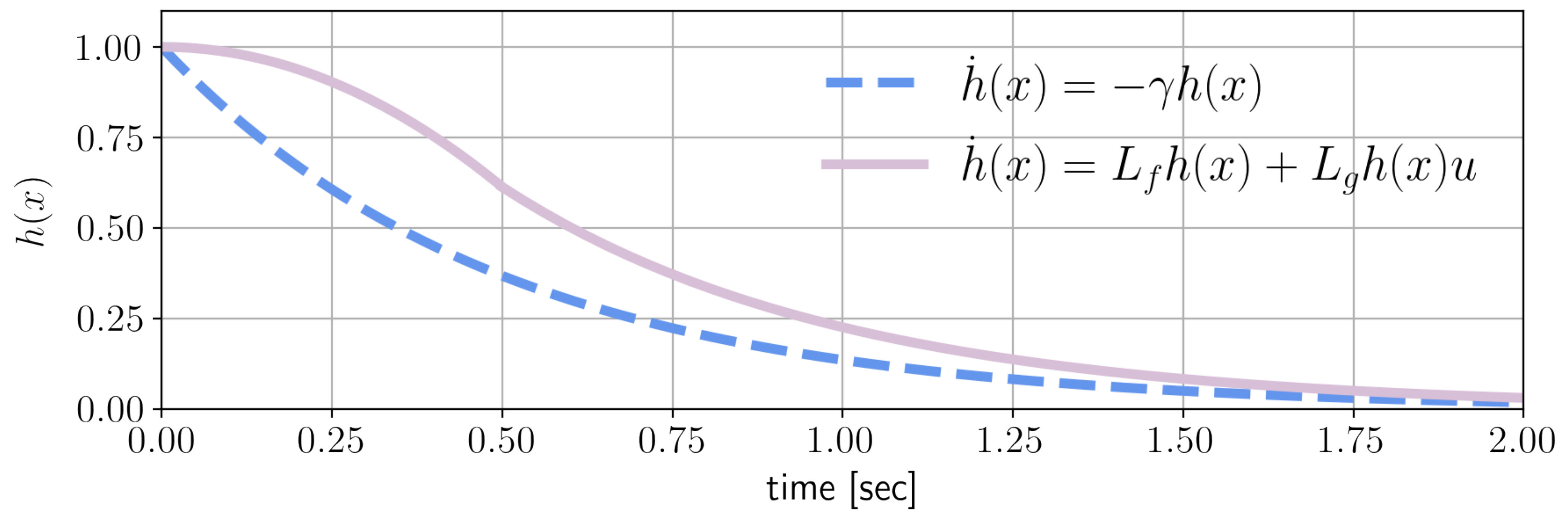
$$\frac{\partial h}{\partial x}\dot{x} + \alpha(h(\mathbf{x})) = L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \alpha(h(\mathbf{x})) \geq 0$$

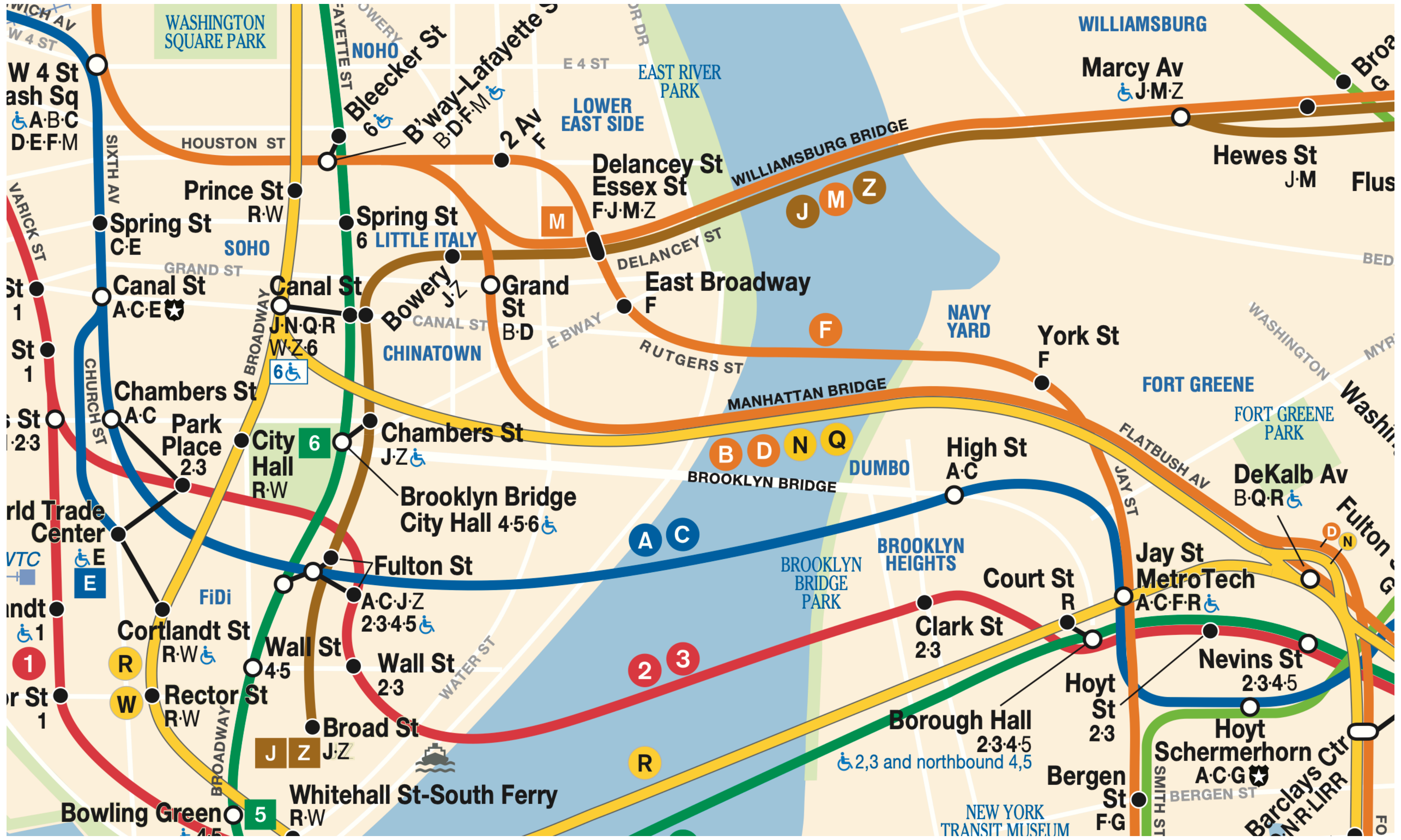
- The function $\alpha(\cdot)$ is a class $\mathscr{K}_\infty$ function.

Ames et al (2019). Control Barrier Functions: Theory and Applications

Control Barrier Function (CBF)

CBF Constraint — Analogy to MTA



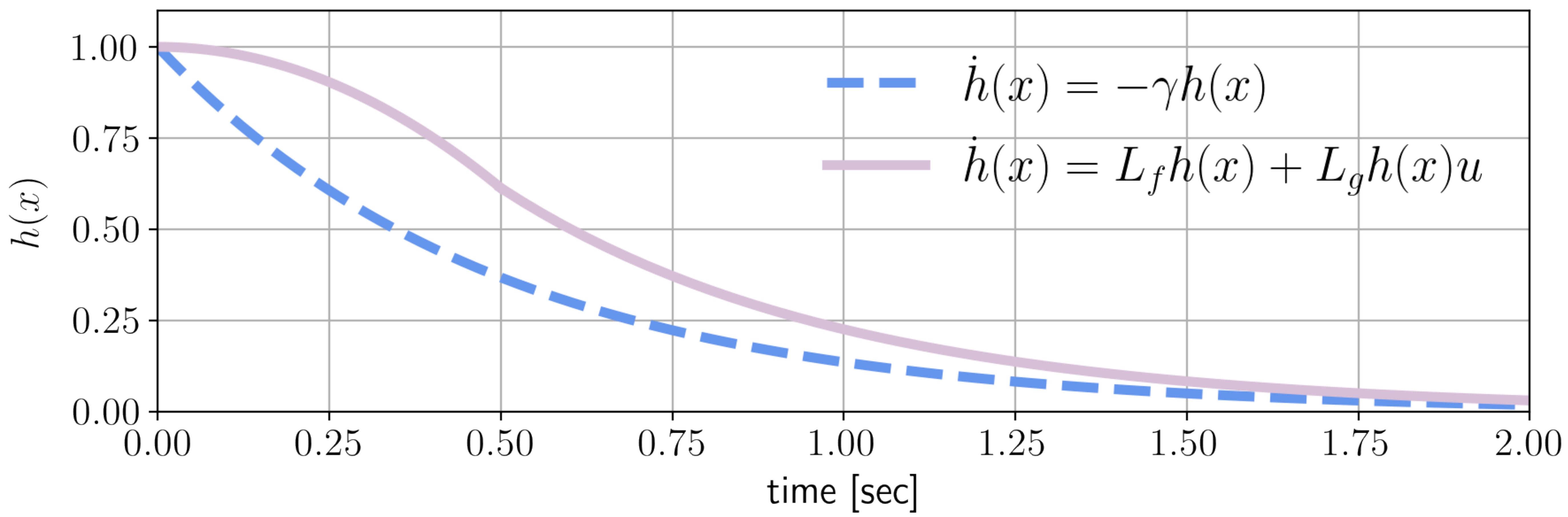


- You want to go to WSQ by taking either A or C train.
- A train is faster or equal to C train between each stop. (Assumption)
- If they start from **Jay St** at the same time, if A never reaches **West 4th**, C with never reach **West 4th**.
- If C reached **West 4th** then A definitely already reached **West 4th**.

Control Barrier Function (CBF)

CBF Constraint

- $L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} \geq -\gamma h(\mathbf{x})$



- Assume that we have two functions:

$\dot{h}(x) = -\gamma h(x)$ and $\dot{\bar{h}}(x) \geq -\gamma \bar{h}(x)$, and further we assume that $\bar{h}(x_0) = h(x_0)$

$$\bar{h}(x_1) = \bar{h}(x_0) + \dot{\bar{h}}(x_0)dt = h(x_0) + \dot{\bar{h}}(x_0)dt \geq h(x_0) + \dot{h}(x_0)dt = h(x_1)$$

- Then it can be concluded that since $\bar{h}(x) \geq h(x)$, and $h(x) = 0$ as time goes to infinity, we have $\bar{h}(x) \geq 0$.

CLF-CBF-QP

Control Lyapunov Function (CLF)

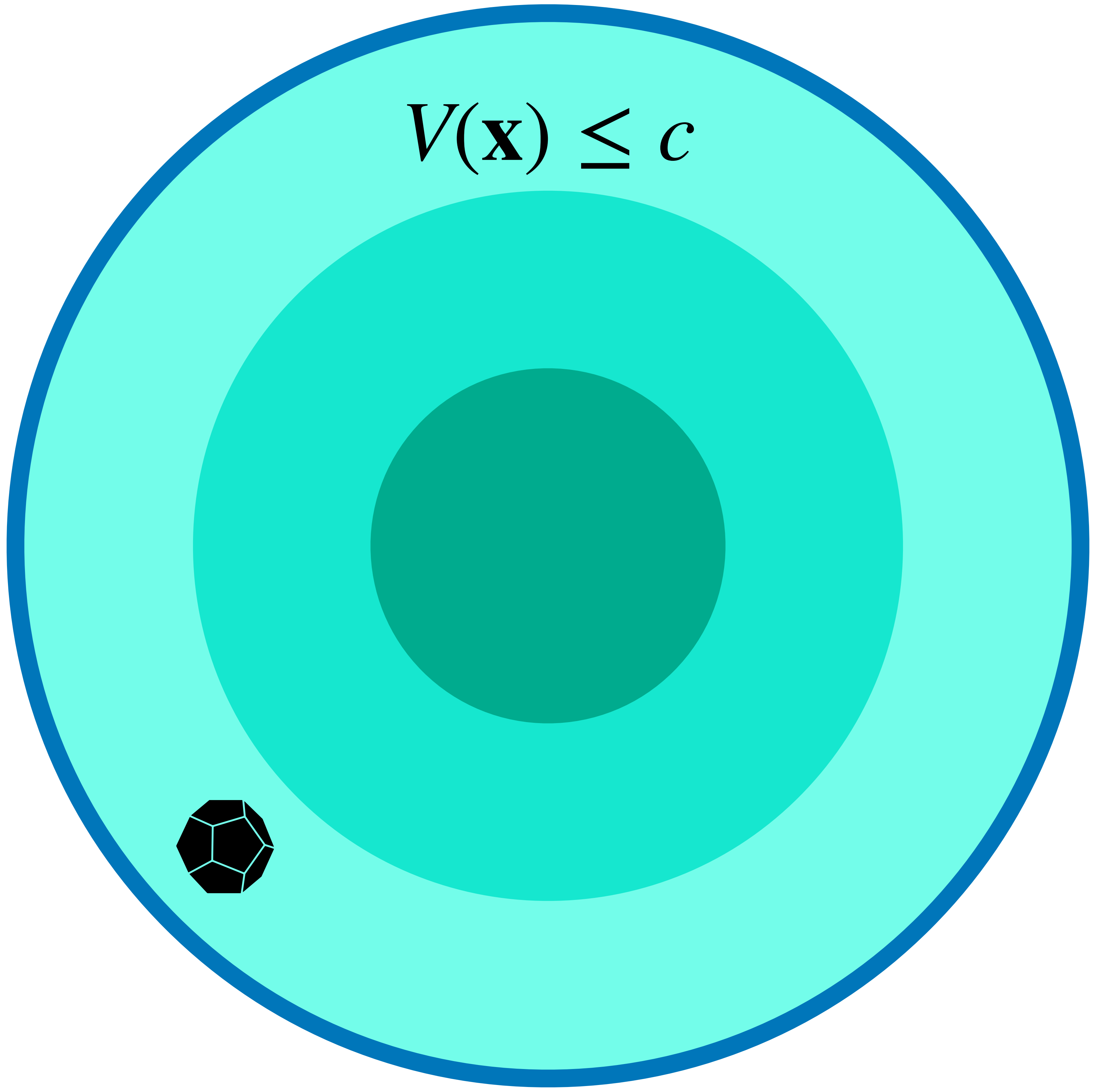
- So far we have been looking at how to perform safe control. Another important quality a controller should possess is stability, i.e. the ability to drive a system from a nonzero state to a region around the origin and stay there.
- And similar to the concept of CBF, if there exist a CLF then the system is stable. A CLF is usually denoted using $V(\mathbf{x})$.

CLF-CBF-QP

Control Lyapunov Function (CLF)

- Some requirements for $V(\mathbf{x})$
  - $\Omega_c := \{\mathbf{x} \in \mathbb{R}^n \mid V(\mathbf{x}) \leq c\}$ is a sub-level set of $V(\mathbf{x})$
  - $V(\mathbf{x}) > 0,\ \forall \mathbf{x} \neq 0$, and $V(\mathbf{0}) = 0$
  - $\dot{V}(\mathbf{x}) \leq 0,\ \forall \mathbf{x} \in \Omega_c \backslash \{\mathbf{0}\}$
- Then we say $V(\mathbf{x})$ is a local control Lyapunov function, and its region of attraction (ROA) is Ω_c. And all of the states within its ROA can be asymptotically stabilized to $\mathbf{0}$

$$\forall x_0 \in \Omega_c,\ \exists u(t),\ \text{s.t.}\ \lim_{t \to \infty} x(t) = \mathbf{0}$$



CLF-CBF-QP

Control Lyapunov Function (CLF)

- Usually we want something faster than asymptotic stability, which is exponential stability.
- This can be achieved by enforcing the following constraint

$$\dot{V}(\mathbf{x}, \mathbf{u}) + \lambda V(\mathbf{x}) \leq 0$$

- Basically, this is saying that we want the CLF to decay faster than an exponential.

CLF-CBF-QP

QP Formulation

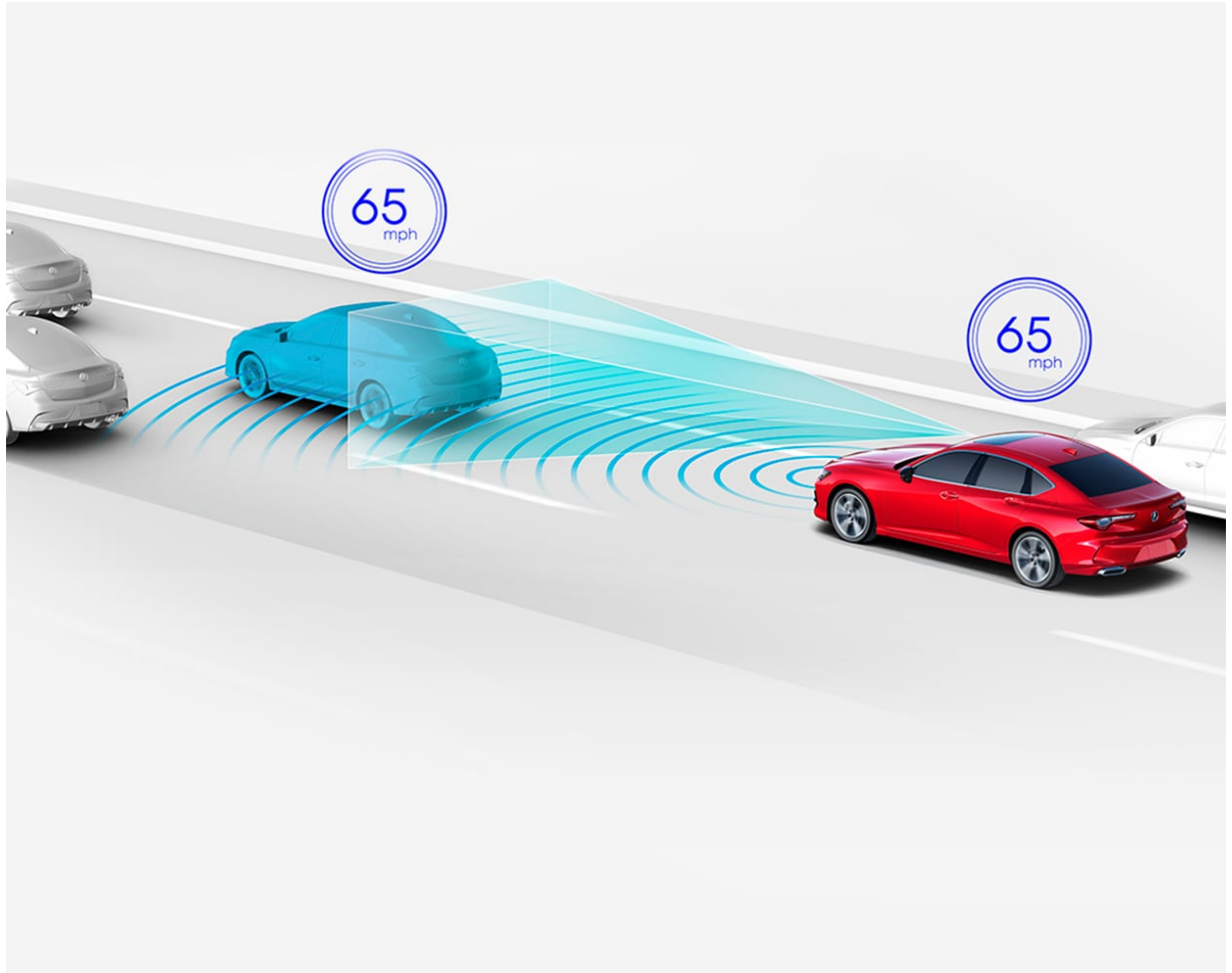
- Here δ is a slack variable that relaxes the CLF constraint

$$
\begin{aligned}
\min_{\mathbf{u},\delta} \quad & \mathbf{u}^T R \mathbf{u} + p\delta^2 \\
\text{subject to} \quad & \mathbf{u} \in \mathscr{U} \\
& L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \gamma h(\mathbf{x}) \geq 0 \\
& L_f V(\mathbf{x}) + L_g V(\mathbf{x})\mathbf{u} + \lambda V(\mathbf{x}) \leq \delta
\end{aligned}
$$

CBF Example

Adaptive Cruise Control

- Maintain a desired velocity while also keeping a safe distance with the leading vehicle.
- This example is borrowed from Jason Choi's guest lecture at UCSD.

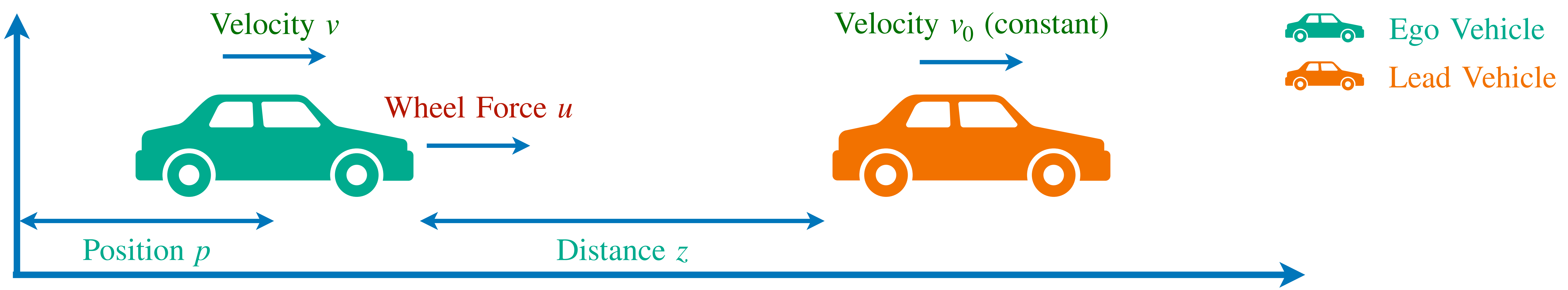


https://www.acura.com/tlx/modals/adaptive-cruise-control-with-low-speed-follow

Ames et al (2014). Control barrier function based quadratic programs with application to adaptive cruise control

CBF Example

Adaptive Cruise Control — Problem Setup



- Dynamics:

$$\begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v \\ -\frac{1}{m}F_r(v) \\ v_0 - v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} u$$

- $F_r(v) = f_0 + f_1 v + f_2 v^2$ is the rolling resistance

- Input constraints: $-mc_d g \le u \le mc_a g$
- Stability Objective: $v \to v_d$ (v_d: desired velocity)
- Safety Objective: $z \ge T_h v$ (T_h: lookahead time)

CBF Example

Adaptive Cruise Control — Formulate CBF for $z \geq T_h v$

- One obvious choice of the CBF is $h(\mathbf{x}) = z - T_h v$, then we have the CBF constraint as

$$\dot{h}(\mathbf{x}, u) + \gamma h(\mathbf{x}) = \frac{T_h}{m}(F_r(v) - u) + (v_0 - v) + \gamma(z - T_h v) \geq 0$$

- If we neglect the effect of the rolling resistance and assuming we are applying the maximum force $u = -c_d m g$, we have

$$\dot{h}(\mathbf{x}, u) + \gamma h(\mathbf{x}) = T_h c_d g + v_0 - v + \gamma(z - T_h v) \geq 0$$

CBF Example

Adaptive Cruise Control — Formulate CBF for $z \geq T_h v$

$$\dot{h}(\mathbf{x}, u) + \gamma h(\mathbf{x}) = T_h c_d g + v_0 - v + \gamma(z - T_h v) \geq 0$$

- A CBF should be positive for all states in the safe set, which is defined by $z \geq T_h v$. We can see that the above function may be negative if v is large with respect to c_d and v_0.
- Note that the definition of the safe set did not specify an upper bound on the velocity v.
- The situation is when the distance z is larger than $T_h v$, but the vehicle cannot break to the same speed as the lead vehicle v_0 before colliding.

CBF Example

Adaptive Cruise Control — Formulate CBF for $z \geq T_h v$

- A better choice of CBF is to incorporate the distance needed to slow down the vehicle to v_0, i.e. distance > lookahead distance + distance to decelerate.

- And under maximum deceleration, i.e. $u = -c_d mg$, we have the

$$\dot{h}(\mathbf{x}, u) = \frac{1}{m} T_h F_r(v) + T_h c_d g$$

- This value is always positive despite the choice of velocity v.

CBF Example

Adaptive Cruise Control — Parameters

```
dt = 0.02
sim_t = 20
x0 = [0, 20, 100]

params.v0 = 14
params.vd = 24
params.m = 1650
params.g = 9.81
params.f0 = 0.1
params.f1 = 5
params.f2 = 0.25
params.ca = 0.3
params.cd = 0.3
params.Th = 1.8
```

```
params.u_max = params.ca * params.m * params.g
params.u_min = -params.cd * params.m * params.g

params.clf.rate = 5  # λ
params.cbf.rate = 5  # γ

'''
Parameters are from
https://github.com/HybridRobotics/CBF-CLF-Helper/
blob/master/demos/run_cbf_clf_simulation_acc.m
'''
```

CBF Example

Results

Exponential Control Barrier Function (ECBF)

Motivation

- When writing the CBF constraint in the form of

$$L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \alpha(h(\mathbf{x})) \geq 0$$

we need the derivative of the CBF $\dot{\mathbf{x}}$ to be a function of the control $\mathbf{u}$

- This might not always be the case, the most simple example is the double integrator system $\ddot{\mathbf{x}} = \mathbf{u}$, which can be seen as a point mass with acceleration control.

Nguyen et al (2016). Exponential Control Barrier Functions for enforcing high relative-degree safety-critical constraints

Exponential Control Barrier Function (ECBF)

Motivation — Double Integrator System

- Let us revisit what the CBF constraint does:
  - For a safe set $\mathscr{C} = \{\mathbf{x} \mid h(\mathbf{x}) \geq 0\}$, if we find the controls that satisfies

$$L_f h(\mathbf{x}) + \alpha(h(\mathbf{x})) \geq 0$$

then we can ensure that the double integrator system never exits the safe set.

  - If we define $d(\mathbf{x}) = L_f h(\mathbf{x}) + \alpha(h(\mathbf{x}))$, then we can also have

$$d(\mathbf{x}) \geq 0$$

Exponential Control Barrier Function (ECBF)

Motivation — Double Integrator System

- We can view $d(\mathbf{x})$ as the new CBF, since we can have the relationship

$$d(\mathbf{x}) \geq 0 \;\; \rightarrow \;\; h(\mathbf{x}) \geq 0$$

- This means that if we can find a control that ensures $d(\mathbf{x}) \geq 0$, then we can also ensure that $h(\mathbf{x}) \geq 0$.

- We can see $d(\mathbf{x})$ as the new CBF and do what we did for CBFs with relative degree one using another class $\mathscr{K}_\infty$ function $\beta(\,\cdot\,)$

$$\dot{d}(\mathbf{x}) + \beta(d(\mathbf{x})) \geq 0 \;\; \rightarrow \;\; d(\mathbf{x}) \geq 0 \;\; \rightarrow \;\; h(\mathbf{x}) \geq 0$$

Exponential Control Barrier Function (ECBF)

Motivation — Double Integrator System

- Since $d(\mathbf{x}) = \dot{h}(\mathbf{x}) + \alpha(h(\mathbf{x}))$, we can write it as

$$d(\mathbf{x}) = h_{\mathbf{x}}\dot{\mathbf{x}} + \alpha(h(\mathbf{x}))$$

- Then we have its time derivative as

$$\dot{d}(\mathbf{x}) = h_{\mathbf{xx}}\dot{\mathbf{x}}^2 + h_{\mathbf{x}}\ddot{\mathbf{x}} + \frac{d\alpha(h(\mathbf{x}))}{dt} = h_{\mathbf{xx}}\dot{\mathbf{x}}^2 + h_{\mathbf{x}}\mathbf{u} + \frac{d\alpha(h(\mathbf{x}))}{dt}$$

ECBF Example

System Dynamics

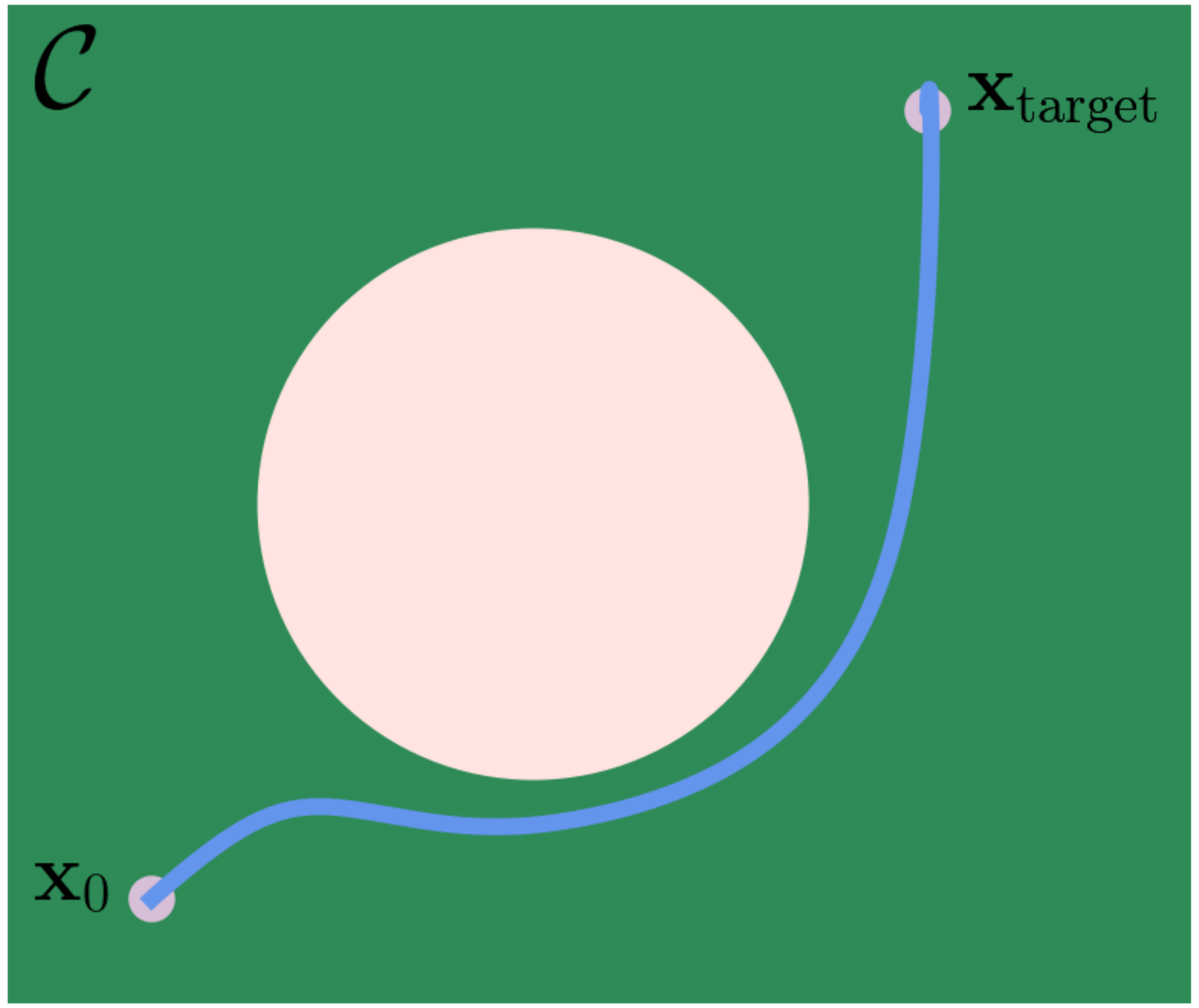
- We use the system dynamics of a double integrator

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

ECBF Example

Find CBF

- The task is to reach a target position without colliding with an obstacle



ECBF Example

ECBF

- We can see a natural choice of the CBF is

$$h(\mathbf{x}) = x^2 + y^2 - r^2$$

- However, since we are controlling the acceleration, the CBF has relative degree two. Thus, an ECBF needs to be used

$$\bar{h} = \dot{h}(\mathbf{x}, \mathbf{u}) + \gamma h(\mathbf{x}) = 2x\dot{x} + 2y\dot{y} + \gamma(x^2 + y^2 - r^2)$$

ECBF Example

CBF-QP

- Assuming that for each state we have a stabilizing controller $\bar{\mathbf{u}} \sim \pi(\mathbf{x})$, then we can write the CBF-QP as

$$\begin{aligned} \min_{\mathbf{u}} \quad & \|\mathbf{u} - \bar{\mathbf{u}}\|^2 \\ \text{subject to} \quad & \mathbf{u} \in \mathcal{U} \\ & L_f h(\mathbf{x}) + L_g h(\mathbf{x})\mathbf{u} + \alpha(h(\mathbf{x})) \geq 0 \end{aligned}$$

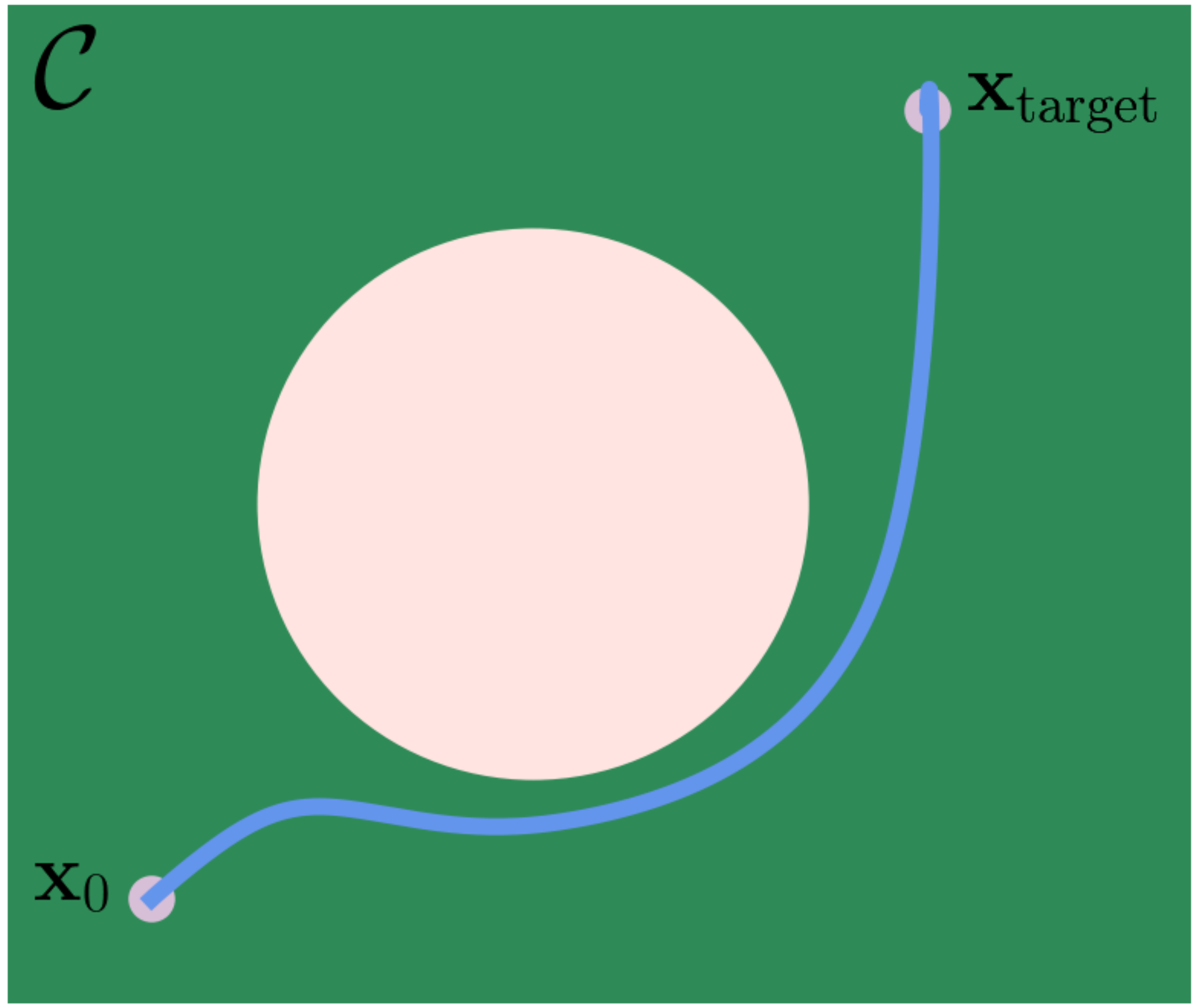
- The system is assumed to be control affine

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}$$

ECBF Example

Results

- Along with a stabilizing controller generated using LQR, we have the following motion.



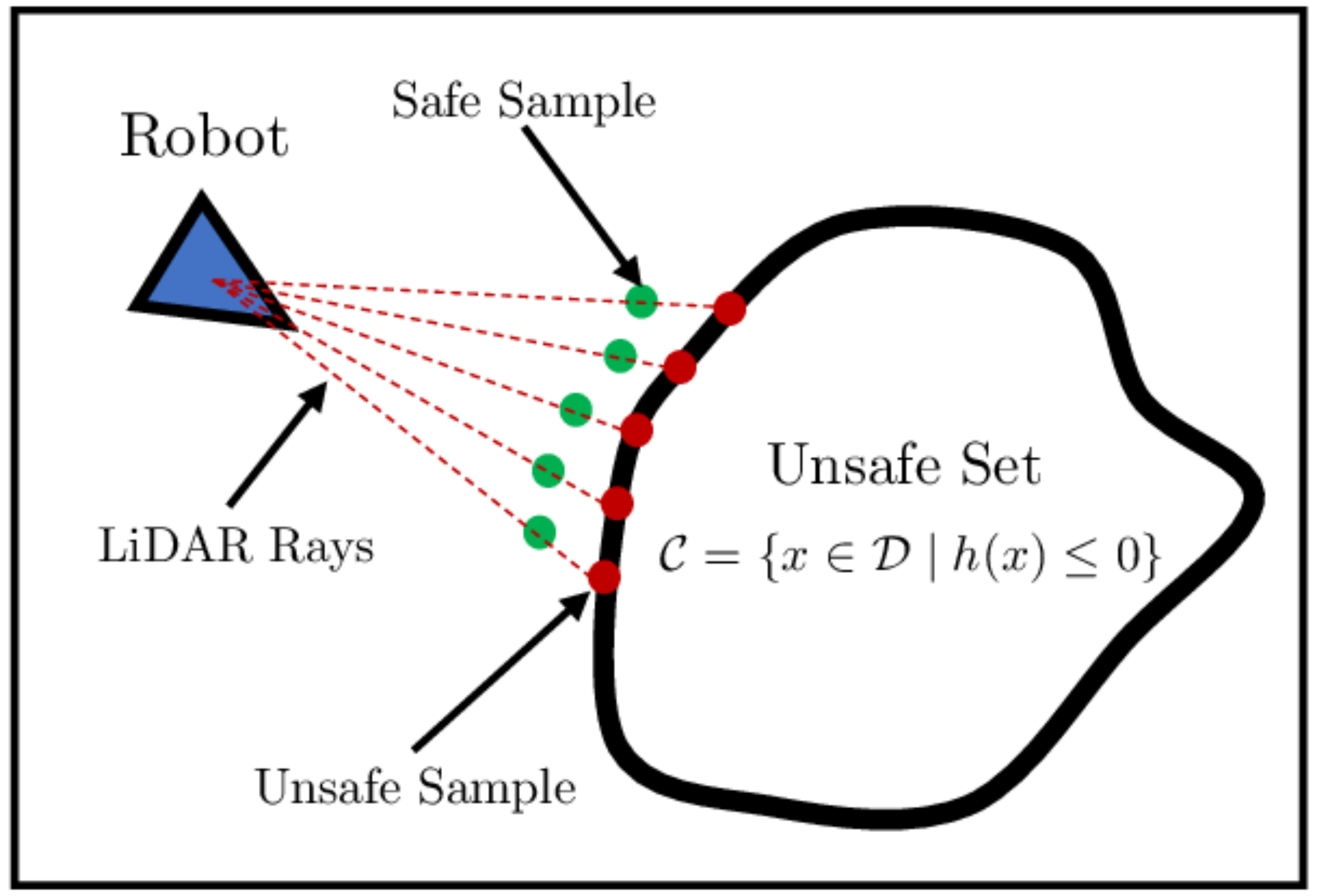
CBF Research

Directions

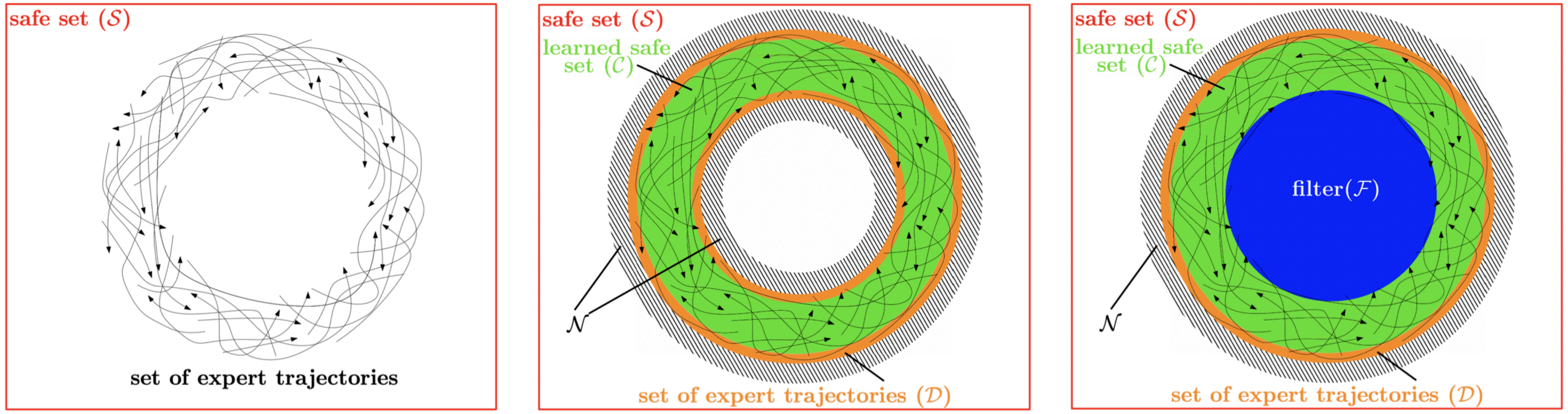
- Synthesis CBFs from data
- CBF with model uncertainty
- CBF for new dynamical systems
- Application to new areas

CBF Research

Synthesis CBFs from Data



Srinivasan et al., IROS 2020



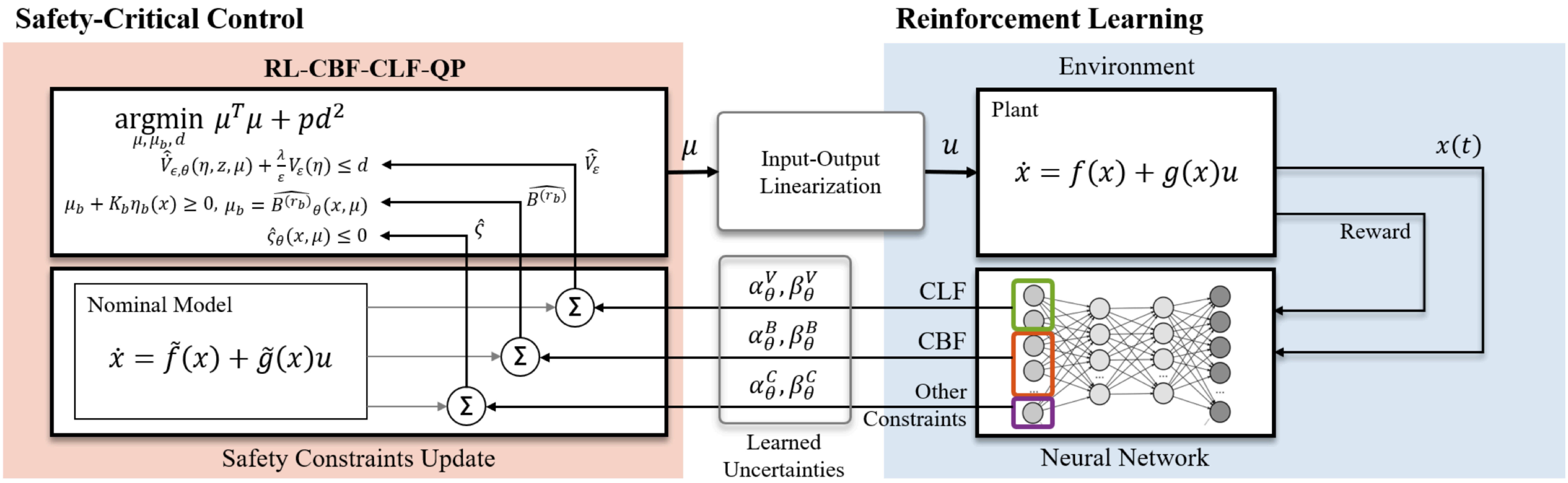
(a) Problem setup. **(b) Desired result.** **(c) Control barrier filter.**

Robey et al., CDC 2020

- *Learning Safe Multi-Agent Control with Decentralized Neural Barrier Certificates*, Qin et al., ICLR 2021

CBF Research

CBF with Model Uncertainty

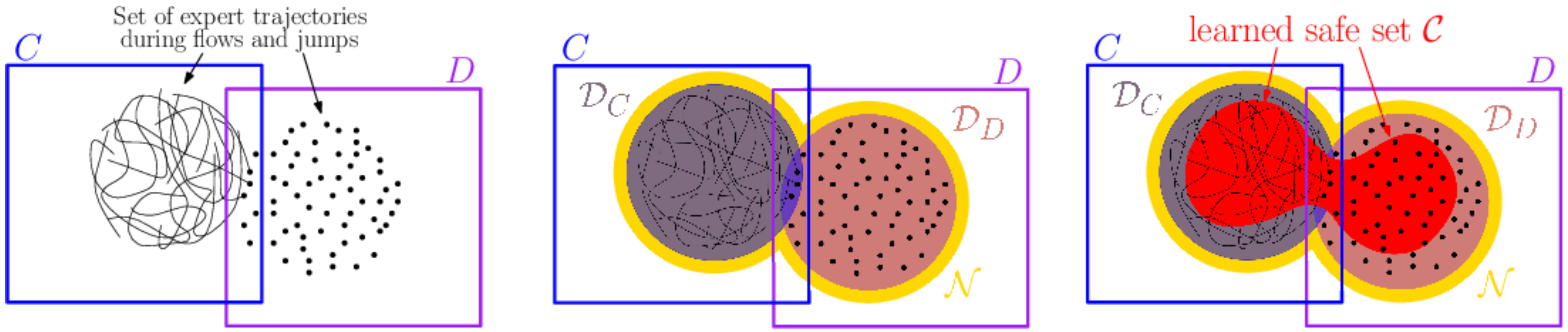


Choi et al., RSS 2020

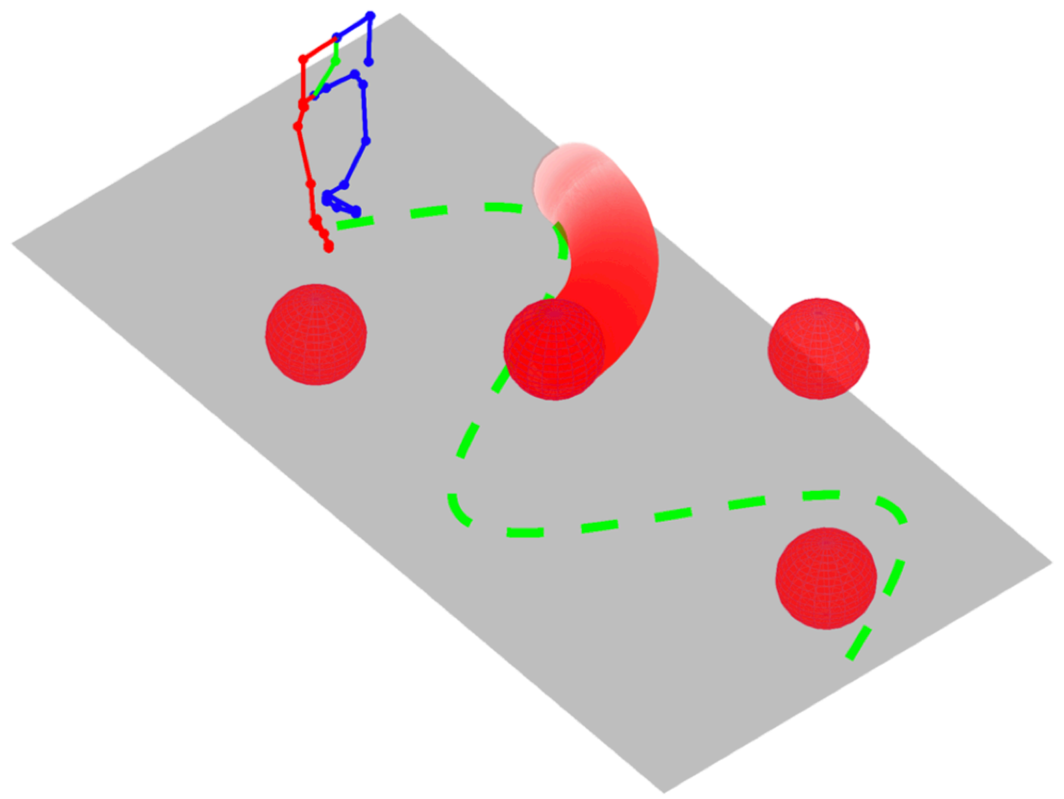
- *Learning for Safety Critical Control with Control Barrier Functions*, Taylor et al., L4DC 2020
- *End-to-End Safe Reinforcement Learning through Barrier Functions for Safety Critical Continuous Control Tasks*, Cheng et al., AAAI 2019

CBF Research

CBF for New Dynamical Systems



Robey et al., CoRL 2020

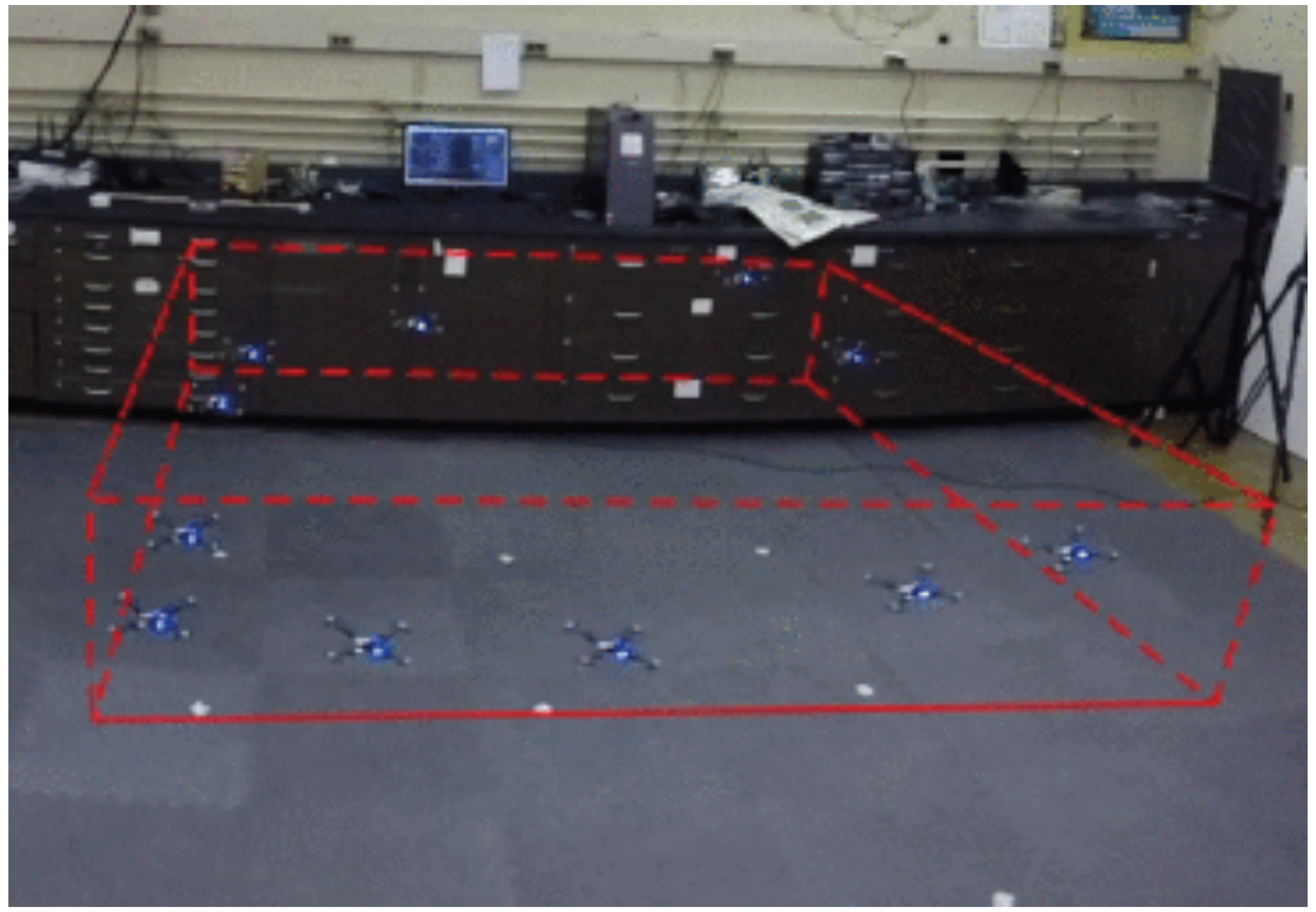


Agrawal et al., RSS 2017

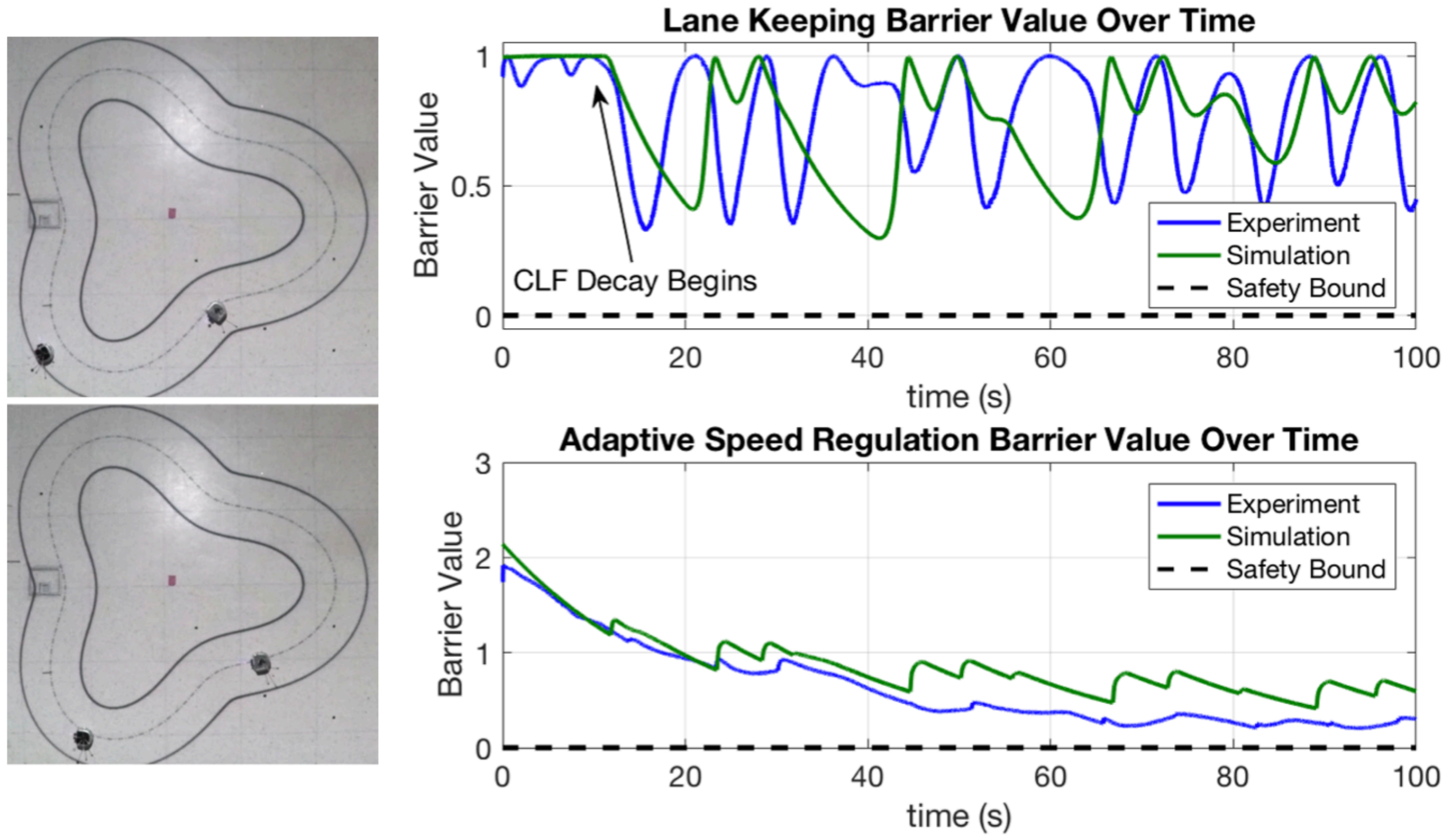
- *Exponential Control Barrier Functions for enforcing high relative-degree safety-critical constraints*, Nguyen et al., ACC 2016

CBF Research

Applications to New Systems



Xu et al., ICRA 2018



Xu et al., CCTA 2017

- *Constraint-driven coordinated control of multi-robot systems*, Notomista et al., ACC 2019